The relationship between the Sérsic law profiles measured along the major and minor axes of elliptical galaxies

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ABSTRACT

In this paper we discuss the reason why the parameters of the Sérsic model best-fitting the major axis light profile of elliptical galaxies can differ significantly from those derived for the minor axis profile. We show that this discrepancy is a natural consequence of the fact that the isophote eccentricity varies with the radius of the isophote and present a mathematical transformation that allows the minor axis Sérsic model to be calculated from the major axis model, provided that the elliptical isophotes are aligned and concentric and that their eccentricity can be represented by a well behaved, though quite general, function of the radius. When there is no variation in eccentricity only the effective radius changes in the Sérsic model, while for radial-dependent eccentricity the transformation, which allows the minor axis Sérsic model to be calculated from the major axis model is given by the Lerch Φ transcendental function. The proposed transformation was tested using photometric data for 28 early-type galaxies.

Key words: galaxies: fundamental parameters – galaxies: photometry – galaxies: structure.

1 INTRODUCTION

It is now recognized that the de Vaucouleurs (1948) $R^{1/4}$ law does not fit the observed light distribution of elliptical galaxies (e.g. Schombergt 1986). A much better representation of the light distribution in bright and dwarf elliptical galaxies and the bulges of spiral galaxies is provided by the Sérsic (1968) law:

$$\log \left( \frac{I(R)}{I_n} \right) = -b_n \left( \frac{R}{R_n} \right)^{1/n} - 1 \tag{1}$$

where $R_n$ is the radius encircling half the total galaxy luminosity and $I_n$ is the intensity at $R_n$. The coefficient $b_n$ is a function of $n$, which can be approximated by the relation $b_n \approx 2n - 0.327$ (Ciotti 1991).

The shape index $n$, which parametrizes the curvature of the Sérsic model has been shown to correlate with the luminosity and size of the galaxy – brighter and larger galaxies having larger values of $n$ (Caon, Capaccioli & D’Onofrio 1993, subsequently cited as CCD93) – and also, notably, with the central velocity dispersion $\sigma_0$ and the mass of the central supermassive black hole (Graham, Trujillo & Caon 2001a; Graham et al. 2001b).

An important source of uncertainty affecting the determination of parameters of the Sérsic model that best describes the light distribution of a galaxy, is on which axis (major, minor or equivalent) the light profile should be fitted.

CCD93 extensively studied the light profiles of many Virgo cluster E and S0 galaxies by independently fitting Sérsic models to their major and minor axes, finding that in ~40 per cent of the galaxies there were large discrepancies between the Sérsic parameters determined along the major and the minor axes. Such discrepancies were found not only among S0 galaxies, which could be misclassified as E galaxies but also among genuine elliptical galaxies such as the E4 galaxy NGC 4621 and E3 galaxy NGC 4406.

Eccentricity gradients imply that both the major and minor axes cannot be, for example, described by the $R^{1/4}$ model. The long observed ellipticity gradients in elliptical galaxies implies that the $R^{1/4}$ model cannot be universal, but this obvious fact has been largely ignored in the literature.

In this paper we demonstrate that the discrepancy between the major and the minor axes Sérsic models in elliptical galaxies can be accounted for by radial variations of the eccentricity of the isophotes. We also present a mathematical formula that, coupled with the eccentricity profile, permits the transformation of the major axis Sérsic model into the minor axis model, provided that the galaxy has well-behaved isophotes, where the eccentricity varies with radius, but which have the same centre and position angle.

In Section 2 we describe the proposed mathematical transformation, the applicability and validity of which is tested using a sample of galaxies selected from those studied by CCD93, as described in Section 3. In Section 4 we present the fitting method and in Section 5 we analyse and discuss our results.

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2 THE LINK BETWEEN MAJOR AND MINOR AXES SÉRISC PROFILES

A simpler and more convenient representation of the Sérsic law is the form given in CCD93:

\[ \mu(R) = A + B R^{1/n}, \]

where, according to equation (1), \( A = -2.5(b_n + \log I_n) \), \( B = 2.5b_n R^{1/n} \). \( R \) may represent the radial variable along the semi-major axis \( a \), the semi-minor axis \( b \) or the equivalent radius \( \sqrt{ab} \). The differential of the surface brightness profile can then be written as

\[ d\mu(R) = \frac{B}{n} R^{1/n - 1} \, dR. \]

Consider two nearby isophotes where the major and minor axes are, respectively, \( a \) and \( b \) for the inner isophote, and \( a' \) and \( b' \) for the outer one, as sketched in Fig. 1. The surface brightness gradient along the major axis may be written as

\[ \frac{d\mu}{da} = \frac{\mu(a') - \mu(a)}{\Delta a}, \]

with a similar expression holding true for the minor axis \( b \).

From the definition of an isophote, we know that \( \mu(a) = \mu(b) \) and \( \mu(a') = \mu(b') \), so the numerators in the right-hand side of expression (4) and in the equivalent expression for \( b \) are equal, while the denominators \( \Delta a \) and \( \Delta b \) will differ according to the radial behaviour of the eccentricity \( e \).

\[ dB = \left[ e + (b/a) \right] da, \]

where \( \mathcal{F}(a) \) will depend on the eccentricity function \( e(a) \). We discuss the case of constant and variable eccentricity functions in the following sections.

2.1 Constant eccentricity

The simplest case is that of concentric isophotes having constant eccentricity. If the eccentricity \( e = b/a = e_c \) is constant, then we have \( b = e_c a \) and \( dB = e_c \, da \), thus:

\[ \frac{d\mu(b)}{db} = \frac{1}{e_c} \frac{d\mu(a)}{da}. \]

By direct integration of equations (2) and (6), we see that in this case the Sérsic index \( n \) will be the same along the major \( (a) \) and the minor \( (b) \) axes, \( n_a = n_b \), and that the \( B \) coefficients on the major and minor axes are related by: \( B_a = B_b / e_c \). Equation (6) shows that the values of \( B \) obtained from the fits along the major and minor axes should not be considered to be independent of each other, as was implicitly assumed by CCD93 (see Section 4). By analysing the relationship between \( B \) and \( b \) in equations (1) and (2), it can be seen that the effect of \( e \) is to stretch out the relationship between \( B_a \) and \( B_b \) (Fig. 2).

Theoretically, the integration constants should be equal, i.e. \( A_a = A_b \), since \( \mu(a = 0) = \mu(b = 0) \). However, in real cases (e.g. CCD93) this equality is broken by a variety of observational uncertainties and practical constraints (for instance, light profiles are fitted within a surface brightness interval the limits of which differ, in general, on the major and minor axes). Consequently, different values for \( A_a \) and \( A_b \) are obtained when the fitted profile is extrapolated to \( R = 0 \).

The Sérsic model along the minor axis is related to the Sérsic model along the major axis by the equation:

\[ \mu(b) = A_b + \frac{B_b}{e_c} b^{1/n_b}, \]

where \( n_b \) is the major axis Sérsic index.

2.2 Variable eccentricity

In most galaxies, eccentricity is neither constant, nor is it a simple function of the radius. Indeed, no general rules seem to govern the radial variation of \( e \), and it is not clear what the physical significance of this variation is Binney & Merrifield (1998). In cD galaxies, \( e \) generally decreases from the centre outwards, while in other galaxies \( e(R) \) may increase, and sometimes it is found to vary non-monotonically with the radius.

Now, if the eccentricity is a differentiable function \( e = e(a) \), then \( dB = e(a) da + ade \) or, equivalently,

\[ dB = \left[ e(a) + \frac{de}{da} \right] da = \mathcal{F}(a) da. \]

In this case, the minor axis profile may have a very different shape from that of the major axis, depending on the form of \( e(a) \). We have integrated equation (5) for a general case in which \( e(a) \) can be expressed as a function of the form

\[ e(a) = e_0 + (e_1 - e_0) \left( \frac{a}{a_M} \right)^{1/l}, \]

where \( a_M \) is the scalelength where the eccentricity equals \( e_1 \). Depending on \( l, e_0 \) and \( e_1 \), equation (9) may describe radial increasing \((e_0 < e_1)\) or decreasing \((e_0 > e_1)\) eccentricities, with different slopes. From equations (8) and (9) we can derive:

\[ \frac{dB}{da} = e_0 + \left[ 1 + l(e_1 - e_0) \left( \frac{a}{a_M} \right)^{1/l} \right] \frac{d\mu}{da}, \]

from which it follows

\[ \frac{d\mu}{dB} = \frac{e_0 + \left[ 1 + l(e_1 - e_0) \left( \frac{a}{a_M} \right)^{1/l} \right]^{-1}}{d\mu/d} \frac{da}{dB}. \]

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By comparing equations (5) and (11) we obtain \( F(a) \),
\[
F(a) = e_0 + (1 + l)(e_1 - e_0) \left( \frac{a}{M_{10}} \right)^l.
\]  
(12)

We can integrate equation (11) in terms of the transcendental function \( \Phi \) (Gradshteyn 
& Ryzhik 2000, see Appendix A), obtaining:
\[
\mu_L(b) = A_e + \frac{B_e}{e_0 n_d} a^{1/n_e} \Phi \left( 1 - \frac{F(a)}{e_0}; 1; \frac{1}{n_d} \right).
\]  
(13)

The variable \( b \) does not appear explicitly on the right-hand side of this equation; 
in order to compute \( \mu \) at a given distance \( b' \) on the minor axis, we should set the 
variable \( a \) on the right-hand side to that value of \( a' \) for which \( b = e(a')a' \).

Equation (13) shows how the major axis Sérsic law is modulated by the \( \Phi \) function to 
give the minor axis light profile. By comparing it with the Sérsic law for the minor 
axis: \( \mu(b) = A_b + B_b a^{1/n_e} \) (equation 2), we can write:
\[
A_b \iff A_e
\]
\[
B_b \iff \frac{B_e}{e_0 n_d}
\]
\[
\delta b^{1/n_e} \iff a^{1/n_e} \Phi \left( 1 - \frac{F(a)}{e_0}; 1; \frac{1}{n_d} \right).
\]  
(14)

2.3 The equivalent-axis Sérsic profile

The Sérsic law can also be expressed as a function of the equivalent radius, defined as 
\( R_{eq} = \sqrt{ab} \). In the case of constant eccentricity, \( e(a) = e_1 = \) constant, 
equation (6) can be written as
\[
\frac{d\mu}{dR_{eq}} = \frac{1}{\sqrt{e_1}} \frac{d\mu}{da} \quad \frac{d\mu}{dR_{eq}} = \frac{2\sqrt{e(a)}}{e(a) + F(a)} \frac{d\mu}{da},
\]  
(15)
while, for variable eccentricity, equation (11) can be expressed as
\[
\frac{d\mu}{dR_{eq}} = \frac{2\sqrt{e(a)}}{e(a) + F(a)} \frac{d\mu}{da},
\]  
(16)
where \( e(a) \) is given by equation (9) and \( F(a) \) by equation (12). We were not able 
to integrate equation (16).

3 DATA SET USED

We applied the algorithm developed in the previous section to 28 galaxies selected from 
those studied by CCD93. Surface brightness and ellipticity profiles for these objects were 
published by Caon, Capaccioli 
& Rampazzo (1990) and Caon, Capaccioli 
& D’Onofrio (1994). The sample we use covered a wide interval of absolute magnitudes 
(\(-22.43 < M_B < -17.29\)) and included at least one object for each morphological type 
(\(E0 \to E7, dS0 \) and \( S0\)).

The correspondence between the Sérsic model index \( n \) for the 
major \( (a) \) and minor \( (b) \) axis also varied: \( n_a > n_b \) for eight galaxies; 
\( n_a < n_b \) for 17, and \( n_a \approx n_b \) for three. The eccentricity (Fig. A1) 
increased with radius for 12 objects, decreased for another 12 and remained approximately constant for 4. The central parts of the light 
profiles, affected by seeing convolution, were excluded when fitting our eccentricity model (equation 9) to the observed profiles.

3.1 Errors

The photometric uncertainties on the CCD93 B-band surface brightness measurements were 
estimated by Caon et al. (1990), and are shown in fig. 3 of their paper. They can be approximated by the power-law function:
\[
\delta \mu = \alpha \mu^\beta,
\]  
(17)
where \( \delta \mu \) is the error, \( \mu \) the surface brightness in magnitudes, \( \alpha \approx 3.25 \times 10^{-15} \) and \( \beta \approx 9.7 \).

The error in the eccentricity can be estimated by approximating the differentials in 
equation (3) by small variations, i.e. \( d\mu \approx \delta \mu \) and \( dR \approx \delta R \), thus obtaining \( \delta \mu = (B\mu R^{1/n_e}) \). Rearranging the terms 
with the help of equation (17) we can write the fractional error \( \delta R/R \) as
\[
\frac{\delta R}{R} = \frac{n_\alpha \mu^\beta}{B R^{1/n_e}} = \frac{n_\alpha (A + B R^{1/n_e})^\beta}{B R^{1/n_e}},
\]  
(18)
where \( R \) may be the \( a \) or \( b \) variable and the coefficients \( A, B, n \) may refer to the major or minor axis accordingly. Since the eccentricity 
is calculated as the quotient \( b/a \), the fractional uncertainties add to give:
\[
\frac{\delta e}{e} \approx \frac{\delta a}{a} + \frac{\delta b}{b}.
\]  
(19)

For example, in the outer parts \( (a = 296, b = 180 \) arcsec) of NGC 4473 we have \( \delta \mu(a) \approx 0.31, \delta \mu(b) \approx 0.43 \) mag arcsec\(^{-2}, \delta a \approx 0.08 \) and \( \delta b \approx 0.15 \), which yields \( \delta e \approx 0.23 \). For NGC 4406 \( (a = 510, \ b = 330 \) arcsec), \( \delta \mu(a) \approx 0.17, \delta \mu(b) \approx 0.28 \) mag arcsec\(^{-2}, \delta a \approx 0.06 \) and \( \delta b \approx 0.11 \), thus \( \delta e \approx 0.17 \).

4 FITTING METHOD

For each of the 28 galaxies of the sample, a Levenberg–Marquardt algorithm was used to fit 
the minor axis surface brightness profile using the transformed major axis Sérsic law. 
The data for the major and minor axes light profiles are those analysed by CCD93.

The fit was performed for both the approximation of constant eccentricity, and for the more general case of variable eccentricity. 
We use the following notation:
\[
\mu_e = A_e + \frac{B_e}{e_0 n_e} a^{1/n_e}
\]  
(20)
and
\[
\mu_L = A_L + \frac{B_L}{e_0 n_d} a^{1/n_e} \Phi \left( 1 - \frac{F(a)}{e_0}; 1; \frac{1}{n_d} \right).
\]  
(21)
Equation (20) is for constant eccentricity and equation (21) is for variable eccentricity. 

We decided to leave the parameters \( A \) and \( B \) completely free. The parameters \( n_e \) is the 
major axis Sérsic index measured by CCD93, while the parameters \( e_0 \) and \( l \) and the function \( F(a) \) are set by our 
fit to the eccentricity profiles. We noted that, ideally, the values we obtained for \( A_e \) and \( A_L \) should equal \( A_0 \), while the values for \( B_e \) and \( B_L \) should be equal to \( B_0 \) (where \( A_e \) and \( B_e \) are the values measured 
by CCD93.) Thus, the validity of our results, and hence of our proposed method, is determined by how close the above parameters 
are to their expected values.

The parameters obtained by fitting equations (20) and equation (21) to the CCD93 minor axes profiles are listed in Table 1, 
where for comparison we include the parameters found by CCD93.

In Appendix B we present the results of Table 1 in graphical format (Fig. B1); these figures also show the major axis profile. The 
bottom panel shows the residuals between the CCD93 data and our best-fitting model.

In Appendix A we present the fits to the eccentricity profiles (Fig. A1) derived from CCD93 data, the solid lines showing the
least-squares fit of the function given by equation (9) to the data points. For some galaxies, we could not use the parameters obtained by this fit and had to determine them interactively. In fact, the Lerch Φ critical radius $a_\phi$ (Appendix C) must be larger than the largest observed radius, for the Lerch Φ function to converge in the radial interval covered by CCD93 observations. The eccentricity profile parameters are shown in Table 2.

5 THE RESULTS

The analysis of the results shown in Table 1 reveals an overall good agreement between the computed and the expected values.

For 14 of the galaxies, both $A_c$ (the zero point in the constant eccentricity model) and $A_L$ (the zero point in the variable eccentricity model) differ by less than 0.5 mag from the best-fitting $A_c$ values determined by CCD93. For further eight galaxies the difference for both coefficients is less than 1 mag. The galaxies with the greatest discrepancies are NGC 4406, 4374 and 4552 for which $|A_L - A_c| > 1.5$ mag.

As for scalelengths (the $B$ parameters in Table 1), 15 galaxies have $B_c$ and $B_L$ values which both differ by less than 20 per cent from $B_n$, while for eight galaxies the difference is less than 30 per cent, the object with the greatest discrepancy is NGC 4564 for which $|B_c - B_n|/B_n = 0.38$.

Fig. 3 shows how the minor axis Sérsic parameters, derived using our method, correlate well with the major axis parameters, this new correlation is a remarkable improvement over that shown in Fig. 2. The fact that the values of $A_c$, $A_L$, $B_c$ and $B_L$ are close to their expected values ($A_n$ and $B_n$) indicates that our transformed major axis Sérsic models can fit the minor axis light profiles quite well.

These results support our proposal that the differences in the Sérsic model of the major and minor axes can be accounted for by radial variations of the isophotes eccentricity, indeed our model seems to be able to provide a valid mathematical description of the links between major and minor axes light profiles and the eccentricity profile.

There is increasing interest in using the $R^{1/n}$ law to address some issues related to the fundamental plane (FP) of elliptical galaxies (Ciotti, Lanzoni & Renzini 1996; Graham & Colless 1997; Ciotti & Lanzoni 1997), thus an extension of the work presented in our current paper would be to investigate how fitting the Sérsic model on different axes may affect the distribution of galaxies on the fundamental plane. This is because two galaxies with the same major axis light profile, but different eccentricity profiles, can give different values for the index $n$ when the Sérsic model is fitted to their equivalent axis profile. This is because $R_{eq} = \sqrt{ab} = a\sqrt{c(a)}$, which may account for some of the scatter observed in the fundamental plane. A full study of this topic is, however, outside the scope of the present paper.
The case of constant eccentricity, and

ACKNOWLEDGMENTS

Figure 3.

NGC 4660 E3 0.55 0.86 0.82 0.70 130
NGC 4564 E6 0.44 0.61 0.60 1.00 190
NGC 4550 S0(7) 0.39 0.22 0.30 0.78 154
NGC 4478 E2 0.82 0.97 0.88 3.00 77
NGC 4476 S0(5) 0.58 0.91 0.85 0.75 154
NGC 4472 E2 1.00 0.75 0.80 0.16 715
NGC 4473 E5 0.45 0.69 0.60 0.55 330
NGC 4476 S0(5) 0.58 0.91 0.85 0.75 154
NGC 4478 E2 0.82 0.97 0.88 3.00 77
NGC 4486 E0 1.00 0.60 0.85 0.65 550
NGC 4550 S0(7) 0.39 0.22 0.30 0.78 154
NGC 4551 E2 0.68 0.82 0.75 1.00 85
NGC 4552 S0(0) 1.00 0.81 0.88 0.43 300
NGC 4646 E6 0.44 0.61 0.60 0.10 190
NGC 4600 S0(6) 0.62 0.85 0.80 1.00 77
NGC 4621 E4 0.65 0.95 0.90 1.00 360
NGC 4623 E7 0.90 0.22 0.41 0.15 110
NGC 4636 E1 1.00 0.62 0.72 0.39 400
NGC 4649 S0(2) 0.77 0.83 0.82 0.60 640
NGC 4660 E3 0.55 0.86 0.82 0.70 130

Figure 3. The relationship between the major axis parameters from CCD93 ($A_\alpha, R_{\alpha}$) and the parameters $A_\alpha, B_\alpha, A_\ell$, and $B_\ell$ derived in this paper. The scatter observed in Fig. 2 is here greatly reduced.

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REFERENCES


APPENDIX A: ECCENTRICITY PROFILES

APPENDIX B: BRIGHTNESS PROFILES

In Fig. B1 we present the results of Table 1 in graphical format.

APPENDIX C: LERCH $\Phi$ FUNCTION

The Lerch $\Phi$ function (named after Mathias Lerch, 1860–1922) is defined as an infinite series (Gradsteyn & Ryzhik 2000)

$$\Phi(z, a, v) = \sum_{i=0}^{\infty} \frac{z^i}{(v+i)^a}. \quad (C1)$$

where $v + i \neq 0$. In the case studied in equation (13) we have

$$\Phi(1 - \frac{F_a}{e_0} ; 1; \frac{1}{n!}) = \sum_{i=0}^{\infty} \frac{n!}{1 + n i} \left(1 + (1 - \frac{1}{e_0}) \left(\frac{1}{e_0}\right)^i\right).$$

(C2)

In this case ($a = 1$), one of the constraints for $\Phi$ to be finite is that we must have $|z| = |1 - F_a/e_0| < 1$, which corresponds to a critical radius $a_c$ beyond which $\Phi$ is finite, given by

$$a_c = \frac{d\Phi}{|1 + n i (1 - e_0/e_0)|^{1/2}}. \quad (C3)$$

We now may write equation (C2) in terms of $a_c$

$$\Phi(1 - \frac{F_a}{e_0} ; 1; \frac{1}{n!}) = \sum_{i=0}^{\infty} \frac{n!}{1 + n i} \left(\frac{a}{a_c}\right)^{i+1}. \quad (C4)$$

The other constraint is that $1 + n i l \neq 0$ in equation (C4) above, thus $n l \neq \ldots, -2, -1, 0$. When fitting the galaxy eccentricity profiles to equation (9) we must take these constraints into account.

The dependence of the Lerch $\Phi$ function on the $n$ and $l$ parameters is shown in Figs C1 and C2. Fig. C1 shows how $\Phi_L$ changes for values of $n = 1, 3, 5, 7, 9, n$ raising in the direction indicated by the arrow. The solid curves have $l = 0.3$ and the dotted curves have $l = 0.7$. The same is true for Fig. C2, for which we plot the values $l = 1, 1, 1, 1, 1, 1$, the solid curves having $n = 3$ and the dotted curves having $n = 9$. For all cases, $e_0 = 0.9$ and $e_1 = 0.1$. The critical radius $a_c$ beyond which the function diverges should be noted. For example, in Fig. C1 the solid line has $a_c/d\Phi_a = 0.62$ and the dotted lines $a_c/d\Phi_a = 0.55$, cf. Equation (C4) and since $a_c$ does not depend on $n$ all the curves in Fig. C1 have the same critical radius.
Figure A1. Eccentricity profiles. The dotted line shows the observed eccentricity from CCD93 data; the solid line is the least-squares fit of formula (9) to the data.
Figure B1. Surface brightness profiles. Solid and dotted lines represent the CCD93 Sérsic fits to the galaxies major and minor axes profiles, respectively; the short and long dashed lines represent our transformation of the major axis Sérsic law by constant and variable eccentricity, respectively. The bottom panel shows the residuals between the CCD93 data and the best-fitting models, using the same line styles as described above.
Figure B1 – Continued.
Figure C1. The dependence of the Lerch $\Phi$ function on the Sérsic index $n$. The plotted values are $n = 1, 3, 5, 7, 9$ increasing as indicated by the arrow. The solid lines are for $l = 0.3$ and the dotted lines for $l = 0.7$. In both cases $e_0 = 0.9$ and $e_1 = 0.1$.

Figure C2. The dependence of the Lerch $\Phi$ function on the eccentricity parameter $l$. The plotted values are $l = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ increasing as indicated by the arrow. The solid lines are for $n = 3$ and the dotted lines for $n = 9$. In both cases $e_0 = 0.9$ and $e_1 = 0.1$.

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